LINEAR ALGEBRA
Instructor's Manual

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Richard Penney
Purdue University

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INSTRUCTOR'S MANUAL

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS

1.1 The Vector Space of $m \times n$ Matrices

Problems begin on page **16**

True-False Questions

1. T, Let the set be $\mathcal{A} = \{A_1, \ldots, A_n\}$ and the subset be $\mathcal{B} = \{A_1, \ldots, A_k\}$. If $\mathcal{B}$ were dependent, then one iof its elements is a combination of the others; say $A_1 = c_2A_2 + \cdots + c_kA_k$. But then $A_1 = c_2A_2 + \cdots + c_kA_k + 0A_{k+1} + \ldots 0A_n$ which contradicts the independence of $\mathcal{A}$. .

2. F, For example let the dependent be $\{A, 2A\}$ where $A \neq 0$ and the subset be $\{A\}$.

3. F, The same example as in 2 works.

4. T, See the proof in 1.

5. F, For example the set might be $\{A, B, 2B\}$ where $\{A, B\}$ is independent.

6. T, Let the set be $\mathcal{S}$. If $\mathcal{S} = \{\mathbf{0}\}$, then it is dependent by definition. Otherwise $\mathcal{S}$ contains $\{\mathbf{0}, A\}$ where $A \neq \mathbf{0}$, which is dependent since $\mathbf{0} = 0A$.

7. F, It will be dependent unless $X = A_i$ for some i.

8. F, For example the set might be $\{X, A, A, 3A\}$ where $\{X, A\}$ is independent.

9. T Each matrix has a non-zero entry in a position where the others have zeros.

10. F. The third is twice the first plus the second.)

11. T. If A_i is a linear combination of the other A_j, then A_i^t is a linear combination of the other A_j^t with the same cosfficients.

12. F. $\tan^2 x = \sec^2 + 1$.

EXERCISES

1.1.

a) $\begin{bmatrix} -1 & -4 & -7 & -10 \\ 1 & -2 & -5 & -8 \\ 3 & 0 & -3 & -6 \end{bmatrix}$, $[3, 0, -3, -6]$, $\begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 8 \\ 4 & 32 \\ 9 & 72 \end{bmatrix}$, $[9, 72]$, $\begin{bmatrix} 8 \\ 32 \\ 72 \end{bmatrix}$

c) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}$, $[-1, 0]$, $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

1.2.

$$(A+B)+C = \begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & -2 \\ 4 & 3 & 3 \\ 2 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & -2 \\ 4 & 3 & 3 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 6 \\ 8 & 4 & -4 \\ 8 & 6 & 6 \\ 4 & 8 & -2 \end{bmatrix}$$

$$A+(B+C) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 1 & 4 \\ 8 & 3 & -2 \\ 6 & 6 & 5 \\ 1 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 6 \\ 8 & 4 & -4 \\ 8 & 6 & 6 \\ 4 & 8 & -2 \end{bmatrix}$$

1.3. $C = A + B$.

1.4. $D = xA + yB = zC$ for any specifc choice aof a, b, c

1.5.

a)

$$[1,1,4] = [1,1,2] + 2[0,0,1]$$

b)

$$[1,2,3] = [1,0,0] + 2[0,1,0] + 3[0,0,1],$$

c)

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$$

e)

$$\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

f)

$$-3 \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -9 & 3 & -6 \\ 0 & -3 & -12 \end{bmatrix}$$

g)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 0 \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

1.6. $P_2 = P_5 - P_1 - P_3 - P_4$, where P_i is the ith row of P.

1.6. $P_2 = P_5 - P_1 - P_3 - P_4$, where P_i is the ith row of P.

1.7. Let P_i be the ith row: $P_2 = P_5 - P_1 - P_3 - P_4$

1.8. A_1 cannot be a linear combination of A_2, A_3 and A_4 because all such linear combinations will have their $(2,1)$ entry equal to zero.

1.9. Each vector has a nonzero entry in the positions where the other two vectors have zeros.

1.10. Suppose first that $A_3 = xA_1 + yA_2$. Then $[0, 0, 8] = [x, 2x + 5y, 3x + 6y]$. Equating the first two entries shows first that $x = 0$, then that $y = 0$, which is impossible due to the third entry. If $A_2 = xA_1 + yA_3$, then $[0, 5, 6] = [x, 2x, 3x + 8y]$. Equating the first entry shows that $x = 0$, which is impossible due to the second entry. Finally, if $A_1 = xA_2 + yA_3$, $[1, 2, 3] = [0, 5x, 6x + 8y]$ which is impossible due to the first entry.

1.11. See the solution to problem 1.10.

1.12. $[1, -1, 0], [1, 0, 0], [2, -2, 0]$, and $[4, -1, 0]$ all belong to the span. $[1, 1, 1]$ does not because its last entry is nonzero.

1.13. a) $-2X + Y = [1, 1, 4]$ (other answers are possible). b) Let $[x, y, z] = aX + bY = [-a - b, a + 3b, -a + 2b]$ and substitute into $5x + 3y - 2z$. You should get 0. c) Any point $[x, y, z]$ that does not solve the equation $5x + 3y - 2z = 0$ will work—for example' $[1, 1, 1]$.

1.14. Finding elements of the span is easy; any linear combination will work. We note that the sum of the entries of each of the given vectors is zero. The same is true for any element of their span. To see his, let Z be an element of the span so that

$$\begin{aligned} Z &= a[1, 1, -1, -1]^t + b[2, -1, -3, 2]^t + c[1, 3, -2, -2]^t \\ &= [a + 2b + c, a - b + 3c, -a - 3b - 2c, -a + 2b - 2c]^t \end{aligned}$$

The sum of the entries of Z are

$$(a + 2b + c) + (a - b + 3c) + (-a - 3b - 2c) + (-a + 2b - 2c) = 0$$

Thus, any vector such as $[1, 1, 1, 1]^t$ whose entries don't total to zero cannot be in the span.

1.15. The general element Z of the span is

$$Z = a[-1, 2, 1]^t + b[2, 5, 1]^t = [-a + 2b, 2a + 5b, a + b]^t$$

The third entry will equal zero if $a = -b$. For example, we might try $a = 1$, $b = -1$ which makes $Z = [-3, -3, 0]^t$. This, however, does not have its first two entries positive. However, if we let $a = -1$ and $b = 1$, we find that $Z = [3, 3, 0]^t$, which does work.

1.16. No. From the second and third entries $aX + bY$ has positive entries only if both a and b are negative; hence the first entry is negative.

1.17. No. For $aX + bY$ to have only positive entries we require $a - b > 0$ and $-2a + 2b > 0$ which contradict each other.

1.18. Yes. $aX + bY$ has positive entries if, for example, a and b are positive and $a > b$.

1.19. a) $f(x) = 2 + \sin x$, b) $f(x) = 2 + \cos x$, c) No. If $f(x) = a\cos x + b\sin x$ then $f(0) = a$ and $f(\pi) = -a$ would both be positive which is impossible.

1.20. a) $aX + 0Y$ for any $a \in R$. b) $aX + bY$ for any non-zero $a, b \in R$ for which $4a + 3b = 0$. c) Since the intersection of two planes through the origin is a line, the span of $\{X, Y\}$ must be a line. Hence let $X = [x, y, x]$ where $z \neq 0$ any $Y = cX$ where $c \neq 1$.

1.21. For $s, t \in \mathbb{R}$ let $sX + tY = Z = [x_3, y_3, z_3]$. Then

$$\begin{aligned} ax_3 + by_3 + cz_3 &= a(sx_1 + tx_2) + b(sy_1 + ty_2)_c(sz_1 + tz_2) \\ &= s(ax_1 + by_1 + cz_1) + t(ax_2 + by_2 + cz_2) = 0 \end{aligned}$$

1.22.

In 1.16, $a - b = 0$ and $a - c = 0$ so $a = b = c \neq 0$. In 1.17, $a - 2b + 4c = 0$ and $-a + 2b + 3c = 0$. Hence $c = 0$ and $a = 2b \neq 0$. The constants exist because every plane has a normal vector.

1.23. For the first part, use various values of a, b, and c in $aX + bY + cZ$. For the second part note that for all scalars a, b and c the $(2, 1)$ entry of $aX + bY + cZ$ is zero. Hence any matrix W in $M(2, 2)$ such that $W_{2,1} \neq 0$ will not be in the span.

1.24.

a) The line containing the origin and the point $(1, 2)$.

b) All of $\mathbb{R}^2$.

c) No. From part b) it appears that if A and B are independent elements of $M(1, 2)$ then any other element of $M(1, 2)$ will be a linear combination of them.

d) The plane containing the two given vectors.

e) $[1, 2, 1] = [1, 1, 0] + [0, 1, 1]$. Hence both $[1, 2, 1]$ and $[0, 1, 1]$ belong to the plane from part e) and the planes in parts d) and e) are the same.

f) The span of these vectors is the line through the origin containing each of them. Two independent vectors will span a plane but two dependent vectors will span a line.

g) The span is the line through $[0, 0, 0]$ containing $[1, 1, 1]$. The span of two linearly dependent vectors is a line through the origin.

1.25. Let V and W be elements of the span. Then $V = aX + bY$ and $W = cX + dY$. Then for $s, t \in \mathbb{R}$, $sV + tW = (as + ct)X + (bs + dt)Y$ which belongs to the span of X and Y.

1.26. Let the columns of A be A_i, $i = 1, 2, 3$. Then $3A_3 - A_2 = A_1$.

1.27. Let the rows of A be A_i, $i = 1, 2, 3$. Then $A_1 = 2A_2 + 2A_3$.

1.28. It is not possible. If the second row is a multiple of the first then

$$A = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$$

Let the columns of A be A_i, $i = 1, 2$. If $a = b = 0$, then the set of columns is $\{\mathbf{0}\}$, which is a dependent set. Otherwise either $A_1 = ab^{-1}A_2$ or $A_2 = ba^{-1}A_1$. The argument for the case where the first row is a multiple of the second is similar.

1.29. Let one row be a linear combination of the other rows. This is easily done keeping all entries non-zero.

1.31. a) Yes: $D = 5A - 2B$. b) Yes: $B = A - C$ so $D = 3A + 2C$. c) Nothing. Given A and B, dependent or not, let $C = A - B$ and $D = 2A + B + 3C$.

1.32. a) Yes: $D = A-B+3(A-B)$. b) Yes: $D = A-B+3C = A-(C-A)+3C$. c) Nothing. Given A and C, dependent or not, let $B = A-C$ and $D = A-B+3C$.

1.33.

a) $119\left(\frac{1}{3}(3\sin^2 x) - \frac{1}{5}(-5\cos^2)\right) = 119$

b) $\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{4}(2e^x) - \frac{1}{6}(3e^{-x})$

c) $-\sinh x + \cosh x = \frac{1}{2}(-e^x + e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = e^{-x}$,

d) From the double angle formula for the cosine function $\cos(2x) = -\sin^2 x + \cos^2 x$.

e) From the double angle formula for the cosine function $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$.

f) $(x+3)^2 = x^2 + 6x + 9$.

g) $x^2 + 3x + 3 = 3(x+1) + \frac{1}{2}(2x^2)$

h) From the angle addition formulas for the sine and cosine functions

$$\sin(x + \frac{\pi}{4}) = \sin(\pi/4)\cos x + \cos(\pi/4)\sin x = \frac{\sqrt{2}}{2}(\cos x + \sin x)$$
$$\cos(x + \frac{\pi}{4}) = \cos(\pi/4)\cos x - \sin(\pi/4)\cos x = \frac{\sqrt{2}}{2}(\cos x - \sin x)$$
$$\sin x = \frac{1}{\sqrt{2}}\left(\sin(x + \frac{\pi}{4}) - \cos(x + \frac{\pi}{4})\right)$$

i)

$$\begin{aligned}\ln\left[(x^2+1)^3/(x^4+7)\right] &= 3\ln(x^2+1) - \ln(x^4+7)\\ &= 3\ln(x^2+1) - 2\ln\sqrt{x^4+7}.\end{aligned}$$

1.34. The span is the set of polynomials of degree $d \leq 2$. Any pair of such polynomials answers the first question.

1.35. The span is the set of polynomials of degree $d \leq 3$. Any pair of such polynomials answers the first question.

1.36. a) Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} x+a & y+b \\ z+c & w+d \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Hence, $x + a = x$, $y + b = y$, $z + c = c$, and $w + d = d$, which imply that $x = y = z = w = 0$. Hence, $B = \mathbf{0}$. b) Solved similarly to a).

1.37. See Example 1.4 on page 11 of the text. For example to prove i) we let $X \in M(n,m)$, $X = [x_{ij}]$. For scalars k and l

$$\begin{aligned}(k+l)X &= [(k+l)x_{ij}]\\ &= [kx_{ij} + lx_{ij}]\\ &= [kx_{ij}] + [lx_{ij}] = kX + lX\end{aligned}$$

1.38. In order, we used vector space properties c), e), e), f), g), j).

1.39. We used Proposition 2 and vector space properties h) and g).

$$\begin{aligned}-(2X+3Y) &= (-1)(2X+3Y)\\ &= (-1)(2X) + (-1)(3Y)\\ &= (-2)X + (-3)Y\end{aligned}$$

1.40. The steps are as shown below. The vector space properties used were Step 1: a) and e), Step 2: c) and e), Step 3: b), e), and Proposition 2 on page 14, Step 4: b), e), and g), Step 5: f), Step 6: h), g), Step 7 j).

$$\begin{aligned}-(aX) + (aX + (bY + cZ)) &= -(aX) + \mathbf{0}\\ (-(aX) + aX) + (bY + cZ) &= -(aX)\\ \mathbf{0} + (bY + cZ) &= -1(aX)\end{aligned}$$

$$
\begin{aligned}
bY + cZ &= (-a)X \\
(-a)^{-1}(bY + cZ) &= (-a)^{-1}((-a)X) \\
(((-a)^{-1}b)Y + ((-a)^{-1}c)Z) &= 1X \\
\left(-\frac{b}{a}\right)Y + \left(-\frac{c}{a}\right)Z &= X
\end{aligned}
$$

1.41. The steps are as shown below. The vector space properties used were Step 1: given, Step 2: a) and e), Step 3: c), e), Step 4: b), e), and e), Step 5: b), e).

$$
\begin{aligned}
X + Y &= \mathbf{0} \\
-X + (X + Y) &= -X + \mathbf{0} \\
(-X + X) + Y &= -X \\
\mathbf{0} + Y &= -X \\
Y &= -X
\end{aligned}
$$

1.42. The steps are as shown below. The vector space properties used were j) and i)along with (1.5) on page 12.

$$
\begin{aligned}
X + (-1)X &= (1)X + (-1)X \\
&= (1 + (-1))X \\
&= \mathbf{0}
\end{aligned}
$$

1.1.2 Applications to Graph Theory I

Problems begin on page **24**

Self-Study Questions

1. The matrices for parts a), b), and c) are respectively

$$
\begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}
$$

2. Possible routes are as in Figure 1.1

3. An airline would not have a flight from a given city A to itself.

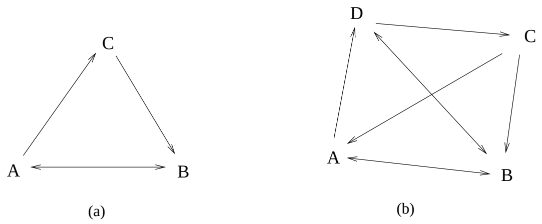

Figure 1.1 Exercise 2.

EXERCISES

1.1. When every city is connected by round trip flights.

1.2. If the jth column is zero then there are no flights into city j. If the ith row is zero, then there are no flights out of city i.

1.3. We list the vertices in the order MGM,MGF,PGM,PGF,F,M,S1,S2,D1,D2.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.4. In a dominance relationship if person $i$dominates person j then person j will not dominate person j and *vice versa*. Hence either a_{ij} and a_{ji} equal 0.

1.5. The route matrix is

$$B = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 3 & 0 & 3 \end{bmatrix}$$

Remark. In Section (3.2.2) on page 179 we show that the two step route matrix is the square of the one step route matrix.

1.6. We list the teams in alphabetical order. The win-loss matrix is

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The number of wins of the jth team is the sume of the entries in the jth row. The number of losses is the sum of the entries in the jth column. Thus team C had 3 wins and one loss.

1.2 Systems

Problems begin on page **36**

In our solutions, Roman numerals refer to equations in a system. Thus, for example "IV" is the fourth equation in a given system.

True-False Questions

1. F. The solution would be a plane if two of the equations are scalar multiples of the first.

2. T. If it has more than one solution, it will have an infinite number of solutions.

3. F. The system would have two free variables. Hence the solution would be a plane.

4. F. The equations might be inconsistent.

5. F. It would have an infinite number of solutions if the equations are dependent.

6. T. The rank is the maximum number of independent equations.

7. a) and b) are false. The planes described by the last equation in each system do not intersect. Hence if system i) is consistent, system (ii) will be inconsistent. c) is false. Translating one of the planes in Figure 1.15 on page 33 could still produce an inconsistent system.

EXERCISES

1.1. X is a solution since $4 \cdot 1 - 2 \cdot 1 - 1 - 1 = 0$ and $1 + 3 \cdot 1 - 2 \cdot 1 - 2 \cdot 1 = 0$. Y is not since $1 + 3 \cdot 2 - 2 \cdot (-1) - 2 \cdot 1 = 7$.

1.2. Let $Z = aX + bY = [a + b, a + 2b, a - b, a + b]$. Substituting Z into the left side of equation I and simplifying produces 0 and substituting Z into the left side of equation II and simplifying produces $7b$. Hence Z is a solution to the system if and only if $b = 0$.

1.3. Let $Z = aX + bY = [a+b, a+b, a+2b, a]^t$. Substituting Z into the left sides of both equation I and equation II and simplifying produces 0. Hence Z is a solution to the system for all a and b.

1.4. Let $Z = aU + bV = [ax+bx', ay+by', az+bz', aw+bw']^t$. Substituting Z into the left sides of both equation I and equation II and simplifying produces 0. Hence Z is a solution to the system for all a and b.

1.5. $Z = aU + bV = [a+b, a+b, 2a+b, -a]^t$. $a+b=1$. Substituting Z into the left side of equation I and simplifying produces $a+b$ and substituting Z into the left side of equation II and simplifying produces $2a+2b$. Hence Z is a solution if and only if $a+b=1$.

1.6. Let $Z = aU + bV = [ax+bx', ay+by', az+bz', aw+bw']^t$. Since U and V satisfy equation I, substituting Z into the left side of both equation I produces

$$a(4x - 2y - z - w) + b(4x' - 2y' - z' - w') = a + b.$$

Similarly substituting Z into the left side of both equation I produces

$$a(x + 3y - 2z - 2w) + b(x' + 3y' - 2z' - 2w') = 2a + 2b.$$

Hence Z is a solution if and only if $a+b=1$.

1.7. In each exercise we give the reduced echelon form of the coefficient matrix followed by the translation vector and the spanning vectors'

a) $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $[2,0]^t$, $[3,1]^t$.

b) $\begin{bmatrix} 1 & 0 & 0 & -59/9 \\ 0 & 1 & 0 & 20/9 \\ 0 & 0 & 1 & 8/9 \end{bmatrix}$, $\frac{1}{9}[-59, 20, 8]^t$, $\mathbf{0}$.

c) $\begin{bmatrix} 1 & 0 & 17/2 & 1 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $[1,0,0]^t$, $\frac{1}{2}[-17,5,2]^t$, 2I + II=III.

d) $\begin{bmatrix} 1 & 0 & 17/2 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, Inconsistent: 2I + II contradicts III.

e) $\begin{bmatrix} 1 & 0 & 0 & 1 & 11 \\ 0 & 1 & 0 & -1 & -6 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$, $[11,-6,2,0]^t$, $[-1,1,-1,1]^t$.

f) $\begin{bmatrix} 1 & 0 & 0 & 10/7 & 11/7 \\ 0 & 1 & 0 & 1/7 & 6/7 \\ 0 & 0 & 1 & -23/14 & 1/7 \end{bmatrix}$, $\frac{1}{7}[11, 6, 1, 0]^t$, $\frac{1}{14}[-20, -2, 23, 2]$.

g) Inconsistent. I + 2II contradicts III.

h) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $-\frac{1}{3}[0, 1, 0]^t$, $[-1, 0, 1]^t$.

i) $\begin{bmatrix} 1 & 0 & 0 & 6/25 \\ 0 & 1 & 0 & -9/25 \\ 0 & 0 & 1 & 7/5 \end{bmatrix}$, $\frac{1}{25}[6, -9, 35]^t$, **0**.

j) $\begin{bmatrix} 1 & 0 & 3/4 & 1 & 5/4 \\ 0 & 1 & 1/4 & 0 & -1/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\frac{1}{4}[5, -1, 0, 0]^t$, $\frac{1}{4}[-3, -1, 4, 0]^t$, $[-1, 0, 0, 1]^t$,

III=4I - II, IV=I + 2II. Since this is a rank 2 system with 4 variables, there are two free variables.

k) $\begin{bmatrix} 1 & 0 & 3/4 & 1 & 0 \\ 0 & 1 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Inconsistent. I + 2II contradicts IV.

l) $\begin{bmatrix} 1 & 0 & 1/3 & -14/3 & 3 \\ 0 & 1 & -2/3 & 4/3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $[3, -1, 0, 0]^t$, $\frac{1}{3}[14, -4, 0, 3]^t$, $\frac{1}{3}[-1, 2, 3, 0]^t$,

III=I+ 2II, IV=7 I+3 II.

1.8. To find the two different solutions choose two different sets of values for the arbitrary parameters, e.g. first make all of them 0 and than make one equal 1 and the rest equal 0.

1.9. For rank 2, let 3 of the equations be linear combinations of 2 given independent equations. For ranks 1 and 3 begin with 1 and 3 independent equations respectively.

1.10. A point (x, y) solves the system if and only if it lies on both lines. Since the lines are parallel, there is no solution to the system.

1.11. The first two lines meet at $(0, 1)$. The third line passes through $(0, a)$. Hence th system is consistent if and only if $a = 1$.

1.2.2 Applications to Circuit Theory

Problems begin on page **43**

Self-Study Questions

1. The new drop is $E = iR = 3 \cdot 7 = 21$ volts.

2. The new drop is $E = iR = 2 \cdot 5 = 10$ volts.

3. The assumed directions are as in Figure 1.2. We obtain the following equations:

Current Law:

$$i_1 = i_3 + i_2 \quad \text{(Nodes C and F)}$$

Voltage Law:

$$\begin{aligned} 0 &= 11 + 3i_3 + 1 \quad \text{(Loop ABCFA)} \\ 0 &= 3i_3 + 1 - 6i_2 \quad \text{(Loop CFEDC)} \end{aligned}$$

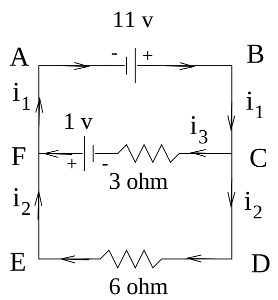

Figure 1.2 Exercise 2 assumed flows.

1.1.

a) We solve the system found in Exercise 3. We find $i_1 = \frac{35}{6}$ amperes from C to B, $i_2 = \frac{11}{6}$ amperes from E to D, $i_3 = 4$ amperes from F to C.

b) The equations are

$$\begin{aligned} i_1 &= i_2 + i_3 \quad \text{(Node C)} \\ 0 &= 5 + 6i_2 \quad \text{(Loop ABCFA)} \\ 0 &= 6i_2 - 10i_3 + 4 \quad \text{(Loop CFEDC)} \end{aligned}$$

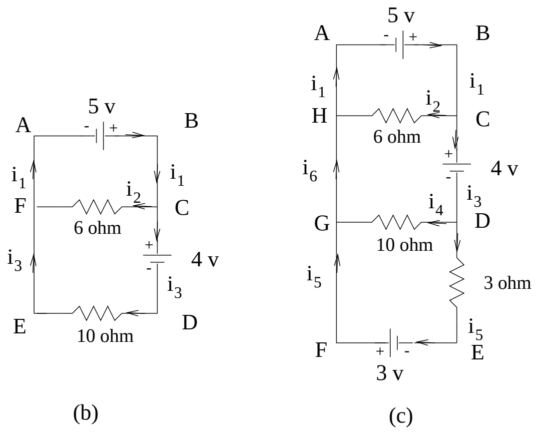

Figure 1.3 Exercises 1.1.b and 1.1.c assumed flows.

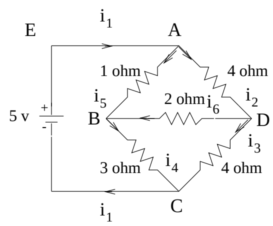

Figure 1.4 Exercise 1.2 assumed flows.

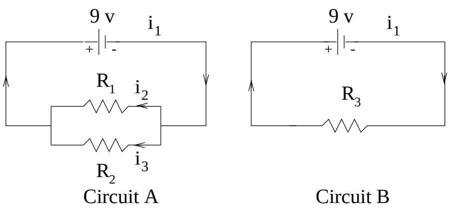

Figure 1.5 Exercise 1.3 assumed flows.

yielding the solution $i_1 = -\frac{14}{15}, i_2 = -5/6, i_3 = -1/10$.

c) The equations are

$$\begin{aligned} i_1 &= i_2 + i_3 \quad \text{(Node C)} \\ i_3 &= i_4 + i_5 \quad \text{(Node C)} \\ 0 &= 5 + 6i_2 \quad \text{(Loop ABCHA)} \\ 0 &= 6i_2 - 10i_4 - 4 \quad \text{(Loop CHGDC)} \\ 0 &= 10i_4 - 3 - 3i_5 \quad \text{(Loop CHGDC)} \end{aligned}$$

yielding the solution $i_1 = -86/15, i_2 = -5/6, i_3 = -49/10, i_4 = -9/10, i_5 = -4$.

1.2. The equations are

$$\begin{aligned} i_1 &= i_2 + i_5 \quad \text{(Node A)} \\ i_2 &= i_6 + i_3 \quad \text{(Node D)} \\ i_4 &= i_6 + i_5 \quad \text{(Node B)} \\ i_1 &= i_4 + i_3 \quad \text{(Node C)} \\ 0 &= 5 + 4i_2 + 4i_3 \quad \text{(Loop EADCE)} \\ 0 &= 5 + i_5 + 3i_4 \quad \text{(Loop EABCE)} \\ 0 &= i_5 - 2i_6 - 4i_2 \quad \text{(Loop ADBA)} \end{aligned} \tag{1.1}$$

yielding the solution

$$\begin{aligned} &i_1 = -295/152, i_2 = -75/152, i_3 = -115/152, \\ &i_4 = -45/38, i_5 = -55/38, i_6 = 5/19 \end{aligned}$$

1.3. Assumming a clockwise flow of current, the equations are

$$\begin{aligned} i_1 &= i_2 + i_3 \\ 0 &= -9 + R_1 i_2 \\ 0 &= -9 + R_2 i_3 \end{aligned}$$

$i_1 = 9\frac{R_2+R_1}{R_1R_2}, i_2 = 9/R_1, i_3 = 9/R_2$.
On the other hand, in Circuit B, the voltage law yields $-9 + i_1R_3 = 0$ so $i_3 = 9/R_3$ showing the equivalence.

1.3 Gaussian Elimination

Problems begin on page **60**

True-False Questions

1. F. One of the rows could represent the equation $0 = 1$.

2. F. If the system is consistent, then there are 3 pivot variables; hence no free variables.

3. T. This is a homogeneous system with more unknowns than equations.

4. F. Four row operations are required.

5. F. Rows III+IV=$[0,0,0,2]$ which corresponds to the inconsistent equation $0 = 2$.

EXERCISES

1.1. a) Neither, b) echelon, c) neither, d) echelon e) reduced echelon.

1.2. We give the solutions followed by the reduced forms.

$$\text{a)} \begin{bmatrix} 0 \\ -4-t \\ -5/2+t \\ 3/2-1/2\,t \\ t \end{bmatrix}, \text{ b)} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \text{ c)} \begin{bmatrix} -3/2 \\ 5/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \text{ d)} \begin{bmatrix} 9-2t \\ 1 \\ t \\ -2 \\ 3 \end{bmatrix}, \text{ e)} \begin{bmatrix} 1-2t \\ 2 \\ t \\ 1 \\ 3 \end{bmatrix}$$

$$\text{a)} \begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{bmatrix}, \text{ b)} \begin{bmatrix} 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{c)} \begin{bmatrix} 3 & 1 & 2 & 6 & 0 \\ 0 & 2/3 & 1/3 & -1 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & -5/2 & -5/4 \end{bmatrix}, \text{ d)} \begin{bmatrix} 1 & 1/2 & 0 & 5 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\text{e)} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

1.3. $$\text{a)} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ b)} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \text{ c)} \begin{bmatrix} 1 & 0 & \frac{10}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix},$$

$$\text{d)} \begin{bmatrix} 1 & 1/2 & 0 & 5 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \text{ e) and f)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

g) $\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$, h) and i) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

j) $\begin{bmatrix} 1 & 0 & -1/3 & -1/3 \\ 0 & 1 & 7/3 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.4. e) , f), h), and i) represent inconsistent systems. Solutions:

a) $\begin{bmatrix} 3-t \\ 1+t \\ t \\ 0 \end{bmatrix}$, b) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, c) $\begin{bmatrix} 10/3 \\ 1/3 \end{bmatrix}$, d) $\begin{bmatrix} -1/2 - t/2 - 5s \\ t \\ 2s \\ s \\ 1 \end{bmatrix}$

g) $\begin{bmatrix} 5 + 1/2\,t \\ -1-t \\ t \\ 2 \end{bmatrix}$, j) $\begin{bmatrix} -1/3 + 1/3\,t \\ 4/3 - 7/3\,t \\ t \end{bmatrix}$

1.5. We give an echelon form for the coefficient matrix followed by the condition:

a) $\begin{bmatrix} 1 & 1 & 2 & b \\ 0 & -1 & -7 & a - 3\,b \\ 0 & 0 & 0 & c - a - 2b \end{bmatrix}$, $c = a + 2b$,

b) $\begin{bmatrix} -1 & -2 & 3 & b \\ 0 & -4 & 20 & c - b \\ 0 & 0 & 35 & a - 5\,b + 2\,r \end{bmatrix}$, No restrictions,

c) $\begin{bmatrix} 2 & -3 & -2 & b \\ 0 & 4 & 7 & -2\,b + a \\ 0 & 0 & 0 & c - a \end{bmatrix}$, $a = c$.

1.6. The right side of I + 2II contradicts III unless $a + 2b = c$.

1.7. The conditions are $3b + c - 2a = 0$ and $d - 3a + 2b = 0$ since an echelon form of the augmented matrix is

$$\begin{bmatrix} 2 & 4 & 1 & 3 & a \\ 0 & 7 & 7/2 & 5/2 & b + 3/2\,a \\ 0 & 0 & 0 & 0 & 3b + c - 2a \\ 0 & 0 & 0 & 0 & d - 3a + 2b \end{bmatrix}$$

1.8. Let A_i, $i = 1, 2, 3, 4$, be the rows of the coefficient matrix. Then $3A_2 + A_3 - 2A_1 = \mathbf{0}$ and $A_4 - 3A_1 + 2A_2 = \mathbf{0}$.

1.9. Let equation (1.30) be written $X = T_1 + tX_1 + sX_2$. As suggested in the hint, replace t by $1 + r + s$ obtaining $X = (T_1 + X_1) + rX_1 + s(X_1 + X_2)$ which is identical with equation (1.29). Conversely, replacing r by $-1 + t - s$ in equation (1.21) results in equation (1.30).

1.10. We give the coefficient matrix, the reduced form, the solution, and the free variable:

a) $$\begin{bmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 2 \\ 1 & 3 & 5 & 11 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 2 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$[x, y, z, w]^t = [5 + t, -6 - 2t, t, 2]^t$, free variable: z.

b) $$\begin{bmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 5 & 2 \\ 1 & 5 & 3 & 11 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1/2 & 0 & 2 \\ 0 & 1 & 1/2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$[x, y, z, w]^t = [2 - t/2, t, -3 - t/2, 2]^t$, free variable: y.

c) In a), $t = z$ while in b) $z = -3 - t/2$. Hence in a) we replace t with $-3 - t/2$ and simplify to get b). In b), $t = y$ while in a) $y = -6 - 2t$. Hence in b) we replace t with $-6 - 2t$ and simplify to get a).

1.11. From the reduced form below, z is still the free variable. The translation vector T is the unique solution with $z = 0$ and the spanning vector is is $Y - T$ where Y is the unique solution with $z = 1$. Hence the expressions for the solutions will be the same.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -6 \\ 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.12.

a) For a line we want one free variable, hence four pivot variables. Thus we need a rank four system of five equations in five unknowns with no zero coefficients. To create such a system begin with a 5×6 matrix that is in echelon form which has exactly four non-zero rows and as few non-zero entries as possible and perform elementary row operations on it until a matrix with all non-zero entries is obtained.

b) For a plane we need two free variables, hence three pivot variables. Thus we need a rank three system of five equations in five unknowns with no zero coefficients. We produce the system exactly as described in a), except we begin with matrix that is in echelon form which has exactly threer non-zero rows.

c) For a line we want one free variable, hence two pivot variables and rank 2. We produce the example as in part a) beginning with a 5×3 rank 2 echelon form matrix.

d) Now we make each equation a multiple of one single equation.

1.13. a) Since $T = \mathbf{0}$, each spanning vector is the solution obtained by setting one free variable equal to 1 and the other equal to 0. Hence the free variables are z and w.

b) $\begin{bmatrix} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

1.14. Each spanning vector should each have a 1 in a position corresponding to one of the free variables while the other spanning vectors have 0's in this position.

1.15. Since $\mathbf{0}$ is a solution to the system, there must be an infinite number of solutions.

1.16. In each part we give the augmented matrix corresponding to the system $B = x_1X_1 + x_2X_2 + x_3X_3$, its reduced form, and our conclusion, which is based upon whether or not the system is consistent.

a) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, in the span,

b) $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 2 & 1 & b \\ -1 & 1 & 1 & c \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -c/2 + a/2 \\ 0 & 1 & 0 & -c/2 - a/2 + b \\ 0 & 0 & 1 & c + a - b \end{bmatrix}$, in the span,

c) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, not in the span.

1.17. Let $x = (b + 2a)/4$ and $y = (b - 2a)/4$.

1.18. This is not hard. Just pick a vector at random. The chances are that it won't be in the span. To prove it, reason as in Exercise 1.16.

1.19. Since there are five variables and four non-zero rows in the reduced form, there must be a free variable.

1.20. Since there are n variables and at most $n-1$ non-zero rows in the reduced form, there must be a free variable.

1.21.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

1.22. Each non-zero row has a 1 in its pivot position and all other rows have a zero in this position.

1.23. a) From the answer to 1.5, No. b) It is the 3×3 identity augmented by a column of constants, c) No. The reduced form has at most two non-zero rows; hence two pivot entries. If consistent, it has a free variable; hence an infinite number of solutions.

1.24. If the rows of A are dependent then A has one of the forms below. In either case the result is clear.

$$A = \begin{bmatrix} a & b \\ ea & eb \end{bmatrix} \quad A = \begin{bmatrix} ec & ed \\ c & d \end{bmatrix}$$

1.25. Our system is equivalent with

$$\begin{aligned} z + w &= -x = -s \\ 2z + w &= -y = -t \end{aligned}$$

which yields $[x, y, z, w]^t = s[1, 0, 1, -2]^t + t[0, 1, -1, 1]^t$.

1.26. No. The equations imply $x = -2(z + w) = -2y$.

1.27. a) We assume that the first column of A is non-zero since this case will take the largest number of flops to reduce. We interchange rows so that $a_{11} \neq 0$ (0 flops), replace a_{1j} by a_{1j}/a_{11} for $2 \leq j \leq n+1$ (n flops), and finally replace a_{11} by 1(0 flops). b) We replace a_{2j} with $a_{2j} - a_{21}a_{1j}$ for $2 \leq j \leq n+1$ ($2n$ flops) and set $a_{2,1} = 0$ (0 flops). c) We do b) $n - 1$ times for a total of

$n + 2n(n-1)$ flops. d) After completing c) we are left with a matrix B of the following form. We may assume that $b_{i2} \neq 0$ for some $i \geq 0$ since we are interested in the maximum number of flops to reduce the matrix to echelon form. We repeat a)-c) for the $(n-1) \times n$ matrix obtained by deleting the first row and column of B. Doing this repeatedly results in the stated formula.

$$B = \begin{bmatrix} 1 & * & \dots & * & * \\ 0 & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \dots & * & * \end{bmatrix} \quad C = \begin{bmatrix} 1 & * & \dots & * & * \\ 0 & 1 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{bmatrix}$$

d) Now we start with an $n \times (n+1)$ matrix C of the form above. Multiplying the last row by a scalar takes one flop since $a \cdot 1 = a$. Subtracting $c_{j,n}$ times this row from the jth row takes 1 flop since $c_{j,n} - c_{j,n} = 0$. Hence eliminating the entries above the final 1 in C requires $2(n-1)$ flops. Thus e) follows by successively repeating this operation for each diagonal entry in C. f) is clear.

1.3.1 Applications to Traffic Flow

Problems begin on page **71**

Self-Study Questions

1. $t + s = 350$.

2. $z + v = 450$.

3. $90 + x = y$.

EXERCISES

1.

a) The equations are $x + w = 20$, $z + w = 50$, $x + y = 50$, $z + y = 80$, and $x + y + z + w = 100$. We list the variables in the order $[x, y, z, w]^t$. The corresponding matrix and its reduced form are respectively:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 20 \\ 0 & 0 & 1 & 1 & 50 \\ 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 80 \\ 1 & 1 & 1 & 1 & 100 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $[x, y, z, w]^t = [20 - w, 30 + w, 50 - w, w]$

b) From the given $6 \leq w \leq 8$. From the solution the largest variable was z which had a value of 54. Thus West Street had the largest traffic flow. The traffic on West Street ranged between 52 and 54 cars per minute.

(a) Our assumptions are equivalent with the inequalities

$$\begin{aligned} &50 - w \geq 20 - w \text{ always true} \\ &50 - w \geq w \Leftrightarrow 25 \geq w \\ &50 - w \geq 30 + w \Leftrightarrow 10 \geq w \end{aligned}$$

Also, since South Street is one way, $w \geq 0$. The claim follows.

c) If West Street is closed then $z = 0$ so $w = 50$ and $x = -30$ which is impossible since East Street is one way.

2. The equations are $y = x + 90$, $y = z + 20$, $w = z + 30$, $w = v + 60$, $v = x + 40$, and $x + y + z + w + v = 120$. We list the variables in the order $X = [x, y, z, w, v]^t$. The corresponding matrix, its reduced form, and the general solution are respectively:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 90 \\ 0 & 1 & -1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & -1 & 60 \\ -1 & 0 & 0 & 0 & 1 & 40 \\ 1 & 1 & 1 & 1 & 1 & 120 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -40 \\ 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[-40 + v, 50 + v, 30 + v, 60 + v, v]^t$$

3.

a) The equations are $z+v = 450$, $350 = z+t$, $u+300 = y+v$, $t+y = 50+s$, $400+x = u+450$, $200+s = x+300$ and $x+y+z+v+s+t+u = 1150$. We list the variables in the order $[x, y, z, v, s, t, u]^t$. The corresponding matrix, its reduced form, and the general solution are respectively:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 350 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 300 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 50 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 50 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 100 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1150 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 125 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 275 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 350 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 225 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 75 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[125, 275 - t, 350 - t, 100 + t, 225, t, 75]$$

b) We need to minimize z. It is smallest when t is largest which occurs for $t = 275$. In this case

$$[x, y, z, v, s, t, u] = [125, 0, 75, 375, 225, 275, 75]^t$$

1.4 Column Space and Nullspace

Problems begin on page **82**

True-False Questions

Note: The dimension of the nullspace of a matrix is the "nullity."

1. T. The rank is at most 3 so the nullity is at least $4 - 3 = 1$.

2. F. If the rank is 3 the nullity would be 0.

3. T. $A[1, 2, 3]^t = A([1, 2, 3]^t - [1, 2, 3]^t) = \mathbf{0}$.

4. T. $A[2, 3, 4]^t = A[1, 1, 1]^t + A[1, 2, 3]^t = [2, 3]^t$.

5. T. If $s = -2$, $X = \mathbf{0}$. Hence $\mathbf{0}$ is a solution and the system is homogeneous.

6. F. $X = (s+2)[1, 2, 1]^t$. Let the system be defined by two independent equations which both are zero at $[1, 2, 1]^t$, e.g. $2x - y = 0$ and $x - y + z = 0$.

7. F. The rank of this system is 1 so it is a line.

8. F. If $X_1 = X_2$, for example, then the spans are equal.

9. F. This set satisfies *none* of the subspace properties.

10. T.

11. F. It not closed under scalar multiplication.

12. F. The set $\{\mathbf{0}\}$ is a one element subspace.

1.1. a) $[0, 5, -11]^t$, b) $[7, 10, 7, 5]^t$, c) $[x_1 + 2x_2 + 3x_3, 4x_1 + 5x_2 + 6x_3]^t$.

EXERCISES

1.3. Compute AX for the general element X in the appropriate $\mathbb{R}^n$ and set each entry of the result equal to a constant. For c) , for example, you might choose

$$x + 2y + 3z = 17$$
$$4x + 5y + 6z = -4$$

1.5. The nullspace is spanned by: a) $\{[-1, 1, 1, 0, 0]^t, [-3, -1, 0, 0, 1]^t\}$ c) $\{[-10, -1, 3]^t\}$, e) $\{[0, 0]\}$.

1.2. In each case we give A and B.

a) $\begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 7 \\ 3 & 10 & 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 7 \\ 4 & 10 & 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, d) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 7 \\ 4 & 10 & 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

e) $\begin{bmatrix} 3 & 7 & 2 \\ 1 & -1 & 1 \\ 5 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, f) $\begin{bmatrix} 2 & -3 & 2 \\ 1 & -6 & 1 \\ -1 & -3 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

g) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$, h) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 2 & -6 & 3 & 2 \\ 5 & -3 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}$

i) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 2 & -6 & 3 & 2 \\ 5 & -3 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 8 \end{bmatrix}$

1.3. Any equation of the form $AX = B$ where A is the given matrix and B is a vector of variables of the appropriate length. For example, for c) we might use

$$\begin{aligned} x + 2y + 3z &= -1 \\ 4x + 5y + 6z &= 2 \end{aligned}$$

1.4. The nullspace of a matrix A is found by computed by augmenting A with a column of zeros and computing its reduced form R which is the reduced form of A augmented with a column of zeros. Each basis element of the nullspace is found by setting one of the free vectors in the system corresponding to R equal to one and the other free variables equal to 0. In each part we give R and the basis for the nullspace.

a) $\begin{bmatrix} 1 & 0 & 0 & -4/7 & 0 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & -6/7 & 0 \end{bmatrix}, \{[4/7, 3/2, 6/7, 1]^t\},$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\{[0,0]^t\}$,

c) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, $\{[1,-2,1]^t\}$

1.5. The process for finding the nullspace is described in the solution to Exercise 1.4. The reduced forms of the matrices R are the matrices in the solution to Exercise 1.3 on page 16 of this manual, augmented by a column of zeros.

a) $\{[-1,1,1,0,0]^t, [-3,-1,0,0,1]^t\}$,
b) $\{[1,-1,0,-1,1]^t\}$,
c) $\{[-10,-1,3]^t\}$,
d) $\{[1/2,0,0,0,-1,1]^t, [-5,0,2,1,0,0]^t, [-1/2,1,0,0,0,0]^t\}$,
e) $\{[0,0]\}$,
f) $\{[0,0]\}$,
g) $\{[-5,1,0,-2,1]^t$, $[1,-2,2,0,0]^t\}$,
h) $\{[0,0,0,0]^t\}$,
i) $\{[0,0,0,0]^t\}$,
j) $\{[1,-4,0,3]^t$, $[1,-7,3,0]^t\}$.

1.6. Let the columns of B be scalar multiples of $[1,2]^t$.

1.7. According to Theorem 1 on page 74 the columns of B must be scalar multiples of $[1,2,3]^t$.

1.8. Let the columns of B be any four vectors which span the same space as the given vectors.

1.9. a) Let the columns of B be any four vectors which span the same space as the given vectors. b) Each element of the span has a zero in the second position.

1.10. Begin with an echelon form 4×6 matrix R that represents a consistent rank three system. Apply elementary row operations to R until a matrix with no zero entries is obtained. This is easiest if R has as few zero entries as possible.

1.11. The zero vector is always a solution. There are an infinity of solutions due to the more unknowns theorem.

1.12. The reduced form R of A is the matrix in (1.20) on page 35 with its fifth column deleted. The nullspace is found by the process described in the solution to Exercise 1.4 above. It is the span of $[-2,1,1,0]^t$ and $[-2,1,0,1]^t$. This exercise demonstrates the translation theorem.

1.13. a) From the reduced form of A given in Example 1.10 on page 53, the general solution is the sum of $[1, 0, 2, 0, 0, 1]^t$ and span of the vectors in b) below.

b) The nullspace is the span of the following vectors: ,
$[1, 1, 0, 0, 0, 0]^t$, $[-2, 0, -1, 1, 0, 0]^t$, $[-1, 0, 1, 0, 1, 0]^t$.

c) One checks by substitution that $[-1, 1, 2, 1, 1, 1]^t$ is a particular solution to the non-homogeneous system. Thus the translation theorem says that the expression in c) is the general solution to the non-homogenous system.

d) Note that

$$[1, 1, 0, 0, 0, 0]^t + [-2, 0, -1, 1, 0, 0]^t = [-1, 1, -1, 1, 0, 0]^t$$

Thus, the three vectors on the right in d) span the the nullspace of A. It follows from the translation theorem that the expression in d) is the general solution to the system.

1.14. a) Let the equation be $ax + by + cz = d$. Since **0** belongs to the span, the zero vector solves the equation, showing that $d = 0$. Substituting $[1, 2, 1]^t$ and $[1, 0, -3]^t$ into the equation yields the system

$$\begin{aligned} a + 2b + \ c &= 0 \\ a \qquad - 3c &= 0 \end{aligned}$$

One solution is $c = 1$, $a = 3$, $b = -2$.

b) Let each equation be a multiple of the one from a).

c) Substitute $[1, 1, 1]^t$ into the system found in b) producing a vector B. The desired system is $AX = B$.

1.15. Let the equation be $ax + by + cz = d$. From the translation theorem, $[1, 2, 1]^t$ and $[1, 0, -3]^t$ must span the solution set for the homogeneous equation. These vectors both solve the equation $3x - 2y + z = 0$. The vector $[1, 1, 1]^t$ is a particular solution to $3x - 2y + z = 2$. Any system in which each equation is a multiple of this equation would have the desired solution set.

1.16. True. $Y_1 = 2X_1 + 2X_2$, $Y_2 = X_1 - X_2$. Hence, Y_1 and Y_2 belong to the span of the X_i. Since spans are subspaces, the span of the Y_i is contained in the span of the X_i. Conversely, $X_1 = \frac{1}{4}Y_1 + \frac{1}{2}Y_2$ and $X_2 = \frac{1}{4}Y_1 - \frac{1}{2}Y_2$ showing that the span of the X_i is contained in the span of the Y_i. Hence, the spans are equal.

1.17. False. Note that $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$ and $Y_3 = X_2$ showing that the span of the Y_i is contained in the span of the X_i. However X_3 is not a linear combination of the Y_i since all of the Y_i have their last two entries equal and the last two entries of X_3 are unequal.

1.18. No, the two answers are not consistent. If the answers were consistent, then the difference of any two solutions to the system would be a solution to the homogeneous system which, form Group I's, answer is spanned by $[-3,1,1]^t$ and $[-1,0,1]^t$. Thus, there should exist s and t such that the following equation is true. The corresponding system is, however, inconsistent.

$$[1,0,0]^t - [1,-1,1]^t = s[-3,1,1]^t + t[-1,0,1]^t$$

1.19. a) For all scalars a, b, c, and d,

$$aX + bY + cZ + dW = (a+3c)X + (b-2d)Y$$

showing that the span of X, Y, Z, and W is contained in the span of X and Y. Conversely,

$$aX + bY = aX + bY + 0Z + 0W$$

showing the equality of the spans.

For span $\{X,Y,Z,W\}$ = span $\{Y,W\}$ we would require $X = cY$ for some scalar c.

1.20. a) If W belongs to span $\{X,Y,Z\}$, then $W = aX + bY + cZ = aX + bY + c(2X+3Y) = (a+2c)X + (b+3c)Y$, which belongs to span $\{X,Y\}$. Conversely, if W belongs to span $\{X,Y\}$, then $W = aX + bY = aX + bY + 0Z$, which belongs to span $\{X,Y,Z\}$. Thus, the two sets have the same elements and are therefore equal.

b) From a) it suffices to prove span $\{X,Y\}$ = span $\{X,Z\}$. This follows from the observations that $Z = 2X + 3Y$ and $Y = \frac{1}{3}Z - \frac{2}{3}X$.

1.21. Let $\mathcal{W}$ satisfy the subspace properties. Then $\mathcal{W}$ is non-empty since it contains the zero vector. If X and Y belong to $\mathcal{W}$ and a and b are scalars, then aX and bY both belong to $\mathcal{W}$. Hence $aX + bY$ also belongs to $\mathcal{W}$ showing that $\mathcal{W}$ is closed under linear combinations. The converse is clear.

1.22. These are all very similar to part a) which is solved in the text.

1.23. (c Let X and Y be elements of $\mathcal{W}$. Then

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad Y = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

where $a+b+c+d = 0 = a'+b'+c'+d'$. Clearly, $\mathcal{W}$ contains the zero vector. If X is as above and k is a scalar, then, $0 = k(a+b+c+d) = ka+kb+kc+kd$ which is equivalent with kX belonging to $\mathcal{W}$. Similarly,

$$\begin{aligned} 0 &= (a+b+c+d) + (a'+b'+c'+d') \\ &= (a+a') + (b+b') + (c'+c') + (d+d') \end{aligned}$$

which is equivalent with $X + Y$ belonging to $\mathcal{W}$. This finishes the proof.

1.24. $\mathcal{W}$ is the first quadrant in $\mathbb{R}^2$. No: $\mathcal{W}$ is not closed under scalar multiplication.

1.25. $\mathcal{W}$ is the integral lattice in $\mathbb{R}^2$. No: $\mathcal{W}$ is not closed under scalar multiplication.

1.26. If the first entry of either X or Y is zero, then $X + Y$ will belong to $\mathcal{W}$. Otherwise, it will not belong to $\mathcal{W}$.

1.27. In this exercise it is easier to use Definition 3 on page 74. Start by noting that the general upper triangular matrix is

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Letting all of the variables equal zero proves that $\mathbf{0}$ is upper-triangular; hence $\mathcal{T}$ is non-empty.

Let A' be another element of $\mathcal{T}$,

$$A' = \begin{bmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{bmatrix}$$

Then for scalars s and t

$$\begin{aligned} sA + tB &= \begin{bmatrix} sa & sb & sc \\ 0 & sd & se \\ 0 & 0 & sf \end{bmatrix} + \begin{bmatrix} ta' & tb' & tc' \\ 0 & td' & te' \\ 0 & 0 & tf' \end{bmatrix} \\ &= \begin{bmatrix} sa + ta' & sb + tb' & sc + tc' \\ 0 & sd + td' & se + te' \\ 0 & 0 & sf + tf' \end{bmatrix} \end{aligned}$$

Hence $sA + tA'$ is upper-triangular, showing that $\mathcal{T}$ is closed undder linear combinations; hence a subspace.

1.28. $X = aA + bB$ will be unipotent if and only if $a + b = 1$. Hence the set of unipotent matrices is not closed under linear combinations and not a subspace.

1.29. $X = aX + bY$ is a solution if and only if and only if $a + b = 1$.

1.30. b) Suppose that y and z are two solutions. Then

$$\begin{aligned} y'' + 3y' + 2y &= \mathbf{0} \\ z'' + 3z' + 2z &= \mathbf{0} \end{aligned}$$

If we add these two equations, we get $(y+z)'' + (y+z)' + 2(y+z) = \mathbf{0}$, showing that $y+z$ is a solution. Multiplying by c yields $(cy)''+(cy)'+2cy = \mathbf{0}$ showing that cy is a solution. The other parts are similar.

1.31. b) $y = \mathbf{0}$ satisfies $y'' + 3y' + 2y = \mathbf{0}$. The sum of any two solutions will solve the equation $y'' + 3y' + 2y = 2t$. Also $z = Cy$ solves $z'' + 3z' + 2z = Ct$. The other parts are similar.

1.32. If $p(x) = p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_n x^n + \cdots + b_0$ then for scalars s and t, $sp(x) + tq(x) = c_n x^n + \cdots + c_0$ where $c_i = sa_i + tb_i$ showing that $\mathcal{P}_n$ is closed under linear combinations.

1.33. The set of all polynomial functions is a subspace but the set of polynomial functions with integral coefficients is not closed under scalar multiplication.

1.34. If f and g satisfy $f(1) = g(1) = 0$ then for scalars s and t, $sf(1) + tg(1) = 0$ showing that $\mathcal{W}$ is closed under linear combinations.

1.35. If f and g satisfy $f'(3) = g'(3) = 0$ then for scalars s and t, $(sf+tg)'(3) = sf'(3) + t'g(3) = 0$ showing that $\mathcal{W}$ is closed under linear combinations.

1.36. If f and g satisfy $f(1) = g(1) = f(2) = g(2) = 0$ then for scalars s and t, the same is true for $sf + tg$ showing that $\mathcal{W}$ is closed under linear combinations.

1.37. b) $a + b = 1$. c) This is not a subspace.

1.38.

a) $f(x) = x - 3/2$.

b) This is clear since for scalars s and t,

$$\int_1^2 (sf(x) + tg(x))\,dx = s\int_1^2 f(x)\,dx + t\int_1^2 g(x)\,dx$$

c) From the preceding formula for $f, g \in \mathcal{V}$, $sf + tg \in \mathcal{V}$ if and only if $s + t = 1$. Hence $\mathcal{V}$ is not a subspace.

1.39. The zero element belongs to $\mathcal{S} \cap \mathcal{T}$. If X and Y belong to $\mathcal{S} \cap \mathcal{T}$, then X and Y both belong to $\mathcal{S}$ so $aX + bY$ belongs to $\mathcal{S}$ for any scalars a and b. Similarly, $aX + bY$ belongs to $\mathcal{T}$; hence to $\mathcal{S} \cap \mathcal{T}$. Thus, $\mathcal{S} \cup \mathcal{T}$ is closed under linear combinations and is a subspace.

1.40. $\mathcal{S} \cup \mathcal{T}$ is a subspace only if either $\mathcal{S} \subset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{S}$. For the proof, suppose that $\mathcal{S}$ is not contained in $\mathcal{T}$. Then $\mathcal{S}$ contains an element S which is not in $\mathcal{T}$. Then for all T in $\mathcal{T}$, $U = S + T$ must be in $\mathcal{S} \cup \mathcal{T}$. But U cannot be in $\mathcal{T}$ since $S = T - U$ and S is not in $\mathcal{T}$. Thus, U belongs to $\mathcal{S}$, proving that $T = U - S$ belongs to $\mathcal{S}$. Hence, $\mathcal{T} \subset \mathcal{S}$.

1.41. The zero element belongs to $\mathcal{S} + \mathcal{T}$. If X and Y belong to $\mathcal{S} + \mathcal{T}$, then $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ where $X_1, X_2 \in \mathcal{S}$ and $Y_1, Y_2 \in \mathcal{T}$. Then $aX + bY =$

$(aX_1 + bX_2) + (aY_1 + bY_2)$ belongs to $\mathcal{S} + \mathcal{T}$ Thus, $\mathcal{S} + \mathcal{T}$ is closed under linear combinations and is a subspace.